The Isomorphism Problem for Rational Group Rings of Metacyclic Groups

Àngel García Blàzquez

Universidad de Murcia

A joint work with **Ángel del Río Mateos**

NCRA VII Conference, 5-7th July

Group Rings: Definition

Definition (RG)

$$RG = \bigoplus_{g \in G} Rg$$

The Isomorphism Problem for Group Rings

$$RG \cong RH \stackrel{?}{\Rightarrow} G \cong H$$

The Isomorphism Problem for Group Rings

$$RG \cong RH \overset{?}{\Rightarrow} G \cong H$$

First Proposed in [Higman 1940].

The Isomorphism Problem for Group Rings

$$RG \cong RH \overset{?}{\Rightarrow} G \cong H$$

First Proposed in [Higman 1940].

$$G, H \text{ abelian}, \ |G| = |H| \Rightarrow \mathbb{C}G \cong \mathbb{C}H$$

The Isomorphism Problem for Group Rings

$$RG \cong RH \overset{?}{\Rightarrow} G \cong H$$

First Proposed in [Higman 1940].

$$G, H \text{ abelian, } |G| = |H| \Rightarrow \mathbb{C}G \cong \mathbb{C}H$$

Fix R, conditions on G, H s.t. $RG \cong RH \Rightarrow G \cong H$

The Isomorphism Problem for Group Rings

$$RG \cong RH \overset{?}{\Rightarrow} G \cong H$$

First Proposed in [Higman 1940].

$$G, H \text{ abelian}, \ |G| = |H| \Rightarrow \mathbb{C}G \cong \mathbb{C}H$$

$$\text{Fix } R, \text{ conditions on } G, H \text{ s.t.} \quad RG \cong RH \Rightarrow G \cong H$$

$$R = \mathbb{Z}, \quad \mathbb{Z}G \cong \mathbb{Z}H \Rightarrow RG \cong RH \text{ for every } R$$

Positive results for $R = \mathbb{Z}$

- Abelian groups [Higman 1940],

Positive results for $R = \mathbb{Z}$

- Abelian groups [Higman 1940],
- Metabelian groups [Whitcomb 1968],

Positive results for $R = \mathbb{Z}$

- Abelian groups [Higman 1940],
- Metabelian groups [Whitcomb 1968],
- p-groups [Roggenkamp and Scott 1987],

Positive results for $R = \mathbb{Z}$

- Abelian groups [Higman 1940],
- Metabelian groups [Whitcomb 1968],
- p-groups [Roggenkamp and Scott 1987],
- Nilpotent groups [Roggenkamp and Scott 1987],

Positive results for $R = \mathbb{Z}$

- Abelian groups [Higman 1940],
- Metabelian groups [Whitcomb 1968],
- p-groups [Roggenkamp and Scott 1987],
- Nilpotent groups [Roggenkamp and Scott 1987],
- Abelian-by-nilpotent groups [§5 de Weiss 1988],

Positive results for $R = \mathbb{Z}$

- Abelian groups [Higman 1940],
- Metabelian groups [Whitcomb 1968],
- p-groups [Roggenkamp and Scott 1987],
- Nilpotent groups [Roggenkamp and Scott 1987],
- Abelian-by-nilpotent groups [§5 de Weiss 1988],
- Simple groups [Kimmerle et al. 1990],

Positive results for $R = \mathbb{Z}$

- Abelian groups [Higman 1940],
- Metabelian groups [Whitcomb 1968],
- p-groups [Roggenkamp and Scott 1987],
- Nilpotent groups [Roggenkamp and Scott 1987],
- Abelian-by-nilpotent groups [§5 de Weiss 1988],
- Simple groups [Kimmerle et al. 1990],
- Supersolvable groups [Kimmerle 1991],

Positive results for $R = \mathbb{Z}$

- Abelian groups [Higman 1940],
- Metabelian groups [Whitcomb 1968],
- p-groups [Roggenkamp and Scott 1987],
- Nilpotent groups [Roggenkamp and Scott 1987],
- Abelian-by-nilpotent groups [§5 de Weiss 1988],
- Simple groups [Kimmerle et al. 1990],
- Supersolvable groups [Kimmerle 1991],
- Frobenius groups [Kimmerle 1991],

Positive results for $R = \mathbb{Z}$

- Abelian groups [Higman 1940],
- Metabelian groups [Whitcomb 1968],
- p-groups [Roggenkamp and Scott 1987],
- Nilpotent groups [Roggenkamp and Scott 1987],
- Abelian-by-nilpotent groups [§5 de Weiss 1988],
- Simple groups [Kimmerle et al. 1990],
- Supersolvable groups [Kimmerle 1991],
- Frobenius groups [Kimmerle 1991],
- Hamiltonian 2-groups [§9 de Milies and Sehgal 2002].

Hertweck's counterexample

In 2001, Martin Hertweck presented a counterexample.

Motivation I

Metabelian groups [Whitcomb 1968]

Motivation I

Metabelian groups [Whitcomb 1968]

Definition

*G **metabelian** if there is $A \trianglelefteq G$ s.t. A and G/A abelian.*

Motivation I

Metabelian groups [Whitcomb 1968]

Definition

*G **metabelian** if there is $A \trianglelefteq G$ s.t. A and G/A abelian.*

G, H s.t. $RG \cong RH$?

Motivation I

Metabelian groups [Whitcomb 1968]

Definition

*G **metabelian** if there is $A \trianglelefteq G$ s.t. A and G/A abelian.*

G, H s.t. $RG \cong RH$ for R field. [Brauer 1963]

Motivation I

Metabelian groups [Whitcomb 1968]

Definition

*G **metabelian** if there is $A \trianglelefteq G$ s.t. A and G/A abelian.*

G, H s.t. $RG \cong RH$ for R field. [Brauer 1963]

[Dade 1971]

Motivation II

Definition

G ***metacyclic*** *if there is* $N \trianglelefteq G$ *s.t.* N *and* G/N *are cyclic.*

Motivation II

Definition

*G **metacyclic** if there is $N \trianglelefteq G$ s.t. N and G/N are cyclic.*

[Baginski 1988], [Sandling 1996], $F_pG \cong F_pH \Rightarrow G \cong H$

Motivation II

Definition

*G **metacyclic** if there is $N \trianglelefteq G$ s.t. N and G/N are cyclic.*

[Baginski 1988], [Sandling 1996], $F_pG \cong F_pH \Rightarrow G \cong H$

Char(0)?

The Isomorphism Problem for Metacyclic Groups

G, H metacyclic:

$$\mathbb{Q}G \cong \mathbb{Q}H \overset{?}{\Rightarrow} G \cong H$$

The Isomorphism Problem for Metacyclic Groups

G, H metacyclic:

$$\mathbb{Q}G \cong \mathbb{Q}H \stackrel{?}{\Rightarrow} G \cong H$$

Work in Progress.

The Isomorphism Problem for Metacyclic Groups

G, H metacyclic:

$$\mathbb{Q}G \cong \mathbb{Q}H \overset{?}{\Rightarrow} G \cong H$$

Work in Progress. OK for nilpotent.

The Isomorphism Problem for Metacyclic Groups

First, p-groups.

The Isomorphism Problem for Metacyclic Groups

First, p-groups.

Abelian p-groups already done: [Perlis and Walker 1950].

Information from $\mathbb{Q}G$

- The size of the group

Information from $\mathbb{Q}G$

- The size of the group
- The abelianizer G/G'

Information from $\mathbb{Q}G$

- The size of the group
- The abelianizer G/G'
- The number of conjugacy classes (=dimension of the center)

Information from $\mathbb{Q}G$

- The size of the group
- The abelianizer G/G'
- The number of conjugacy classes (=dimension of the center)
- The number of conjugacy classes of cyclic subgroups (=number of simple componentes)

Classification of metacyclic p-groups [Hempel 2000]

$$P_{m,n,r,s} = \left\langle a, b \mid a^{p^m} = 1, b^{p^n} = a^{p^s}, a^b = a^{1+p^r} \right\rangle, \begin{cases} 1 \leq r \leq s \leq n, \\ s \leq m \leq r+s \\ \text{if } p=2 \text{ then } r \geq 2. \end{cases}$$

$$N_{m,n,r,s} = \left\langle a, b \mid a^{2^m} = 1, b^{2^n} = a^{2^s}, a^b = a^{-1+2^r} \right\rangle, \begin{cases} \max(r, m-1) \leq s \leq m \leq n+r, \\ 2 \leq \min(n, r) \\ s < n+r-1 \text{ or } m = s. \end{cases}$$

$$D_{2^r} = \left\langle a, b \mid a^{2^{r-1}} = b^2 = 1, a^b = a^{-1} \right\rangle, \text{with } r \geq 3.$$

$$D^+_{2^r} = \left\langle a, b \mid a^{2^{r-1}} = b^2 = 1, a^b = a^{2^{r-2}+1} \right\rangle, \text{with } r \geq 4.$$

$$D^-_{2^r} = \left\langle a, b \mid a^{2^{r-1}} = b^2 = 1, a^b = a^{2^{r-2}-1} \right\rangle, \text{with } r \geq 4.$$

$$Q_{2^r} = \left\langle a, b \mid a^{2^{r-1}} = 1, b^2 = a^{2^{r-2}}, a^b = a^{-1} \right\rangle, \text{with } r \geq 3.$$

Strategy

$$\mathbb{Q}G \Rightarrow \text{ Type of } G???$$

Strategy

$$\mathbb{Q}G \Rightarrow \text{ Type of } G???$$

$$\mathbb{Q}G \Rightarrow G???$$

Strategy

$$\mathbb{Q}G \Rightarrow \text{ Type of } G???$$

$$\mathbb{Q}G \Rightarrow G???$$

We know $|G|$, G/G', NCC and $NCCC$.

Strategy

$\mathbb{Q}G \Rightarrow$ Type of G???

$\mathbb{Q}G \Rightarrow G$???

We know $|G|$, G/G', NCC and $NCCC$.

D_{2^r} and Q_{2^r} already studied

Strategy II

$$D_{2^r} = \left\langle a, b \mid a^{2^{r-1}} = b^2 = 1, a^b = a^{-1} \right\rangle, \text{with } r \geq 3.$$

$$D^+_{2^r} = \left\langle a, b \mid a^{2^{r-1}} = b^2 = 1, a^b = a^{2^{r-2}+1} \right\rangle, \text{with } r \geq 4.$$

$$D^-_{2^r} = \left\langle a, b \mid a^{2^{r-1}} = b^2 = 1, a^b = a^{2^{r-2}-1} \right\rangle, \text{with } r \geq 4.$$

$$Q_{2^r} = \left\langle a, b \mid a^{2^{r-1}} = 1, b^2 = a^{2^{r-2}}, a^b = a^{-1} \right\rangle, \text{with } r \geq 3.$$

Groups of Type P

$$P_{m,n,r,s} = \left\langle a, b \mid a^{p^m} = 1, b^{p^n} = a^{p^s}, a^b = a^{1+p^r} \right\rangle, \begin{cases} 1 \leq r \leq s \leq n, \\ s \leq m \leq r+s \\ \text{if } p = 2 \text{ then } r \geq 2. \end{cases}$$

Groups of Type P

$$P_{m,n,r,s} = \left\langle a, b \mid a^{p^m} = 1, b^{p^n} = a^{p^s}, a^b = a^{1+p^r} \right\rangle, \begin{cases} 1 \le r \le s \le n, \\ s \le m \le r+s \\ \text{if } p = 2 \text{ then } r \ge 2. \end{cases}$$

Theorem

p prime, G and H finite, metacyclic of type P. TFAE:

1. $G \cong H$.
2. $\mathbb{Q}G \cong \mathbb{Q}H$.
3. $|G| = |H|$, $G/G' \cong H/H'$ and $NCCC_G = NCCC_H$.

Groups of type N

$$N_{m,n,r,s} = \left\langle a, b \mid a^{2^m} = 1, b^{2^n} = a^{2^s}, a^b = a^{-1+2^r} \right\rangle, \begin{cases} \max(r, m-1) \leq s \leq m \leq n + r, \\ 2 \leq \min(n, r) \\ s < n + r - 1 \text{ or } m = s. \end{cases}$$

Groups of type N

$$N_{m,n,r,s} = \left\langle a, b \mid a^{2^m} = 1, b^{2^n} = a^{2^s}, a^b = a^{-1+2^r} \right\rangle, \begin{cases} \max(r, m-1) \leq s \leq m \leq n+r, \\ 2 \leq \min(n, r) \\ s < n+r-1 \text{ or } m = s. \end{cases}$$

Problem

Groups of type N

$$N_{m,n,r,s} = \left\langle a, b \mid a^{2^m} = 1, b^{2^n} = a^{2^s}, a^b = a^{-1+2^r} \right\rangle, \begin{cases} \max(r, m-1) \leq s \leq m \leq n + r, \\ 2 \leq \min(n, r) \\ s < n + r - 1 \text{ or } m = s. \end{cases}$$

Problem

$|G|$, G/G', NCC and $NCCC$ are not enough!

Example

$G = N_{3,4,3,3}$, $H = N_{3,4,2,3}$:

Example

$G = N_{3,4,3,3}$, $H = N_{3,4,2,3}$:

$$\mathbb{Q}N_{3,4,3,3} = 4\mathbb{Q} \oplus 2\mathbb{Q}(i) \oplus 2\mathbb{Q}(\zeta_8) \oplus 2\mathbb{Q}(\zeta_{16}) \oplus H(\mathbb{Q}) \oplus M_2(\mathbb{Q}) \oplus H(\mathbb{Q}(\sqrt{2})) \oplus M_2(\mathbb{Q}(i)) \oplus M_2(\mathbb{Q}(\sqrt{2})) \oplus 4M_2(\mathbb{Q}(\zeta_8))$$

$$\mathbb{Q}N_{3,4,2,3} = 4\mathbb{Q} \oplus 2\mathbb{Q}(i) \oplus 2\mathbb{Q}(\zeta_8) \oplus 2\mathbb{Q}(\zeta_{16}) \oplus H(\mathbb{Q}) \oplus M_2(\mathbb{Q}) \oplus 2M_2(\mathbb{Q}(\sqrt{-2})) \oplus M_2(\mathbb{Q}(i)) \oplus 4M_2(\mathbb{Q}(\zeta_8)),$$

Example

$G = N_{3,4,3,3}$, $H = N_{3,4,2,3}$:

$$\mathbb{Q}N_{3,4,3,3} = 4\mathbb{Q} \oplus 2\mathbb{Q}(i) \oplus 2\mathbb{Q}(\zeta_8) \oplus 2\mathbb{Q}(\zeta_{16}) \oplus H(\mathbb{Q}) \oplus M_2(\mathbb{Q}) \oplus H(\mathbb{Q}(\sqrt{2})) \oplus M_2(\mathbb{Q}(i)) \oplus M_2(\mathbb{Q}(\sqrt{2})) \oplus 4M_2(\mathbb{Q}(\zeta_8))$$

$$\mathbb{Q}N_{3,4,2,3} = 4\mathbb{Q} \oplus 2\mathbb{Q}(i) \oplus 2\mathbb{Q}(\zeta_8) \oplus 2\mathbb{Q}(\zeta_{16}) \oplus H(\mathbb{Q}) \oplus M_2(\mathbb{Q}) \oplus 2M_2(\mathbb{Q}(\sqrt{-2})) \oplus M_2(\mathbb{Q}(i)) \oplus 4M_2(\mathbb{Q}(\zeta_8)),$$

$$|G| = |H|, \quad G/G' \cong H/H'$$

$NCC_G = NCC_H$ and $NCCC_G = NCCC_H$

$\mathbb{Q}G \cong \mathbb{Q}H \Rightarrow G \cong H$ for p-groups

Theorem

G,H metacyclic finite p-groups. Then,

$$\mathbb{Q}G \cong \mathbb{Q}H \iff G \cong H$$

The general case. Hempel Classification 2001

$$\pi(G) = \langle p \mid |G| \text{ prime s.t. the Hall } p'\text{-subgroup is normal} \rangle$$

The general case. Hempel Classification 2001

$$\pi(G) = \langle p \mid |G| \text{ prime s.t. the Hall } p'\text{-subgroup is normal} \rangle$$

Sylow implies Hall

The general case. Hempel Classification 2001

$$\pi(G) = \langle p \mid |G| \text{ prime s.t. the Hall } p'\text{-subgroup is normal} \rangle$$

Sylow implies Hall

If G solvable, Hall subgroups exist and are conjugated.

The general case. Hempel Classification 2001

$$\pi(G) = \langle p \mid |G| \text{ prime s.t. the Hall } p'\text{-subgroup is normal} \rangle$$

Sylow implies Hall

If G solvable, Hall subgroups exist and are conjugated.

p' means $\{q \mid |G| : q \neq p\}$.

The general case. Hempel Classification 2001

$$\pi(G) = \langle p \mid |G| \text{ prime s.t. the Hall } p'\text{-subgroup is normal} \rangle$$

Sylow implies Hall

If G solvable, Hall subgroups exist and are conjugated.

p' means $\{q \mid |G| : q \neq p\}$.

$G = G_{\pi'} \rtimes G_{\pi}$.

The general case. Hempel Classification 2001

$$\pi(G) = \langle p \mid |G| \text{ prime s.t. the Hall } p'\text{-subgroup is normal} \rangle$$

Sylow implies Hall

If G solvable, Hall subgroups exist and are conjugated.

p' means $\{q \mid |G| : q \neq p\}$.

$G = G_{\pi'} \rtimes G_{\pi}$.

G nilpotent iff $G = G_{\pi}$.

The general case II

$$\mathbb{Q}G \Rightarrow \pi(G)??$$

p-components

Definition (p-component of $\mathbb{Q}G$)

A p-component if $deg(A) = p^*$ *and* $\mathcal{Z}(A) \subseteq \mathbb{Q}(\zeta_{p^m})$, $m > 0$.

$\mathbb{Q}G \Rightarrow \pi(G)$

Lemma

G metacyclic, $F_2 = G_2' G_2^2$, *the Frattini subgroup:*

- $2 \in \pi(G)$ *and*

$$\mathbb{Q}G_2 \cong \bigoplus_{2-components} A$$

- $2 \neq p$,

$$p \in \pi(G) \iff |G_p| \cdot [G_2 : F_2] = \sum_{p-component} dim(A)$$

In this case,

$$[G_2 : F_2]\mathbb{Q}G_p \cong \bigoplus_{p-components} A$$

Proof of the result for nilpotent groups

Theorem

G,H metacyclic finite groups s. t. $\mathbb{Q}G \cong \mathbb{Q}H$, then $\pi(G) = \pi(H)$ and for every $p \in \pi(G)$ we have $G_p \cong H_p$.

The general nilpotent case

Theorem

G, H finite nilpotent metacyclic. Then,

$$\mathbb{Q}G \cong \mathbb{Q}H \iff G \cong H$$

The End

Thank you very much for your attention. I will be waiting for your questions during the break.